CHAPTER 3

VIBRATION THEORY

SINGLE DEGREE OF FREEDOM SYSTEMS (SDOF)

Many simple structures can be idealized as a concentrated or lumped mass, ***m***, supported by a massless structure with stiffness, ***k***, in the lateral direction. A one story building or structural frame as shown in Fig 3.1 has a heavy and stiff/rigid roof. When this building is subjected to lateral load, ***F***, it has only one degree of dynamic freedom, the lateral sway or displacement, ***Δ***, as indicated on the figure. An equivalent dynamic model of this building consists of a single column with equivalent stiffness, *k*, supporting a lumped mass of magnitude, ***m***.

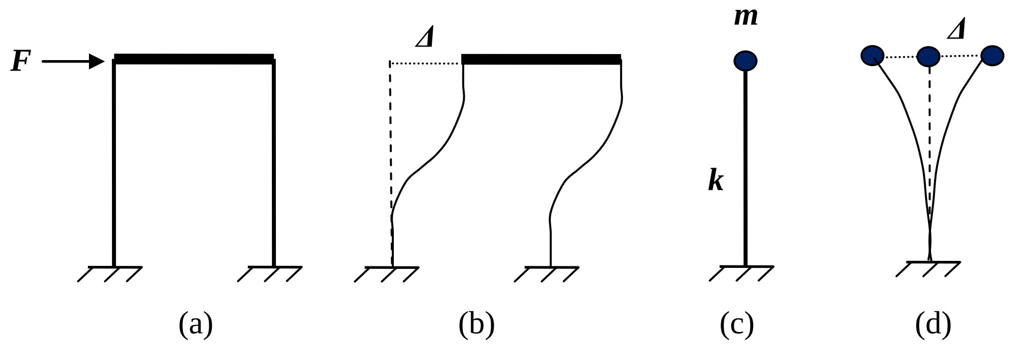


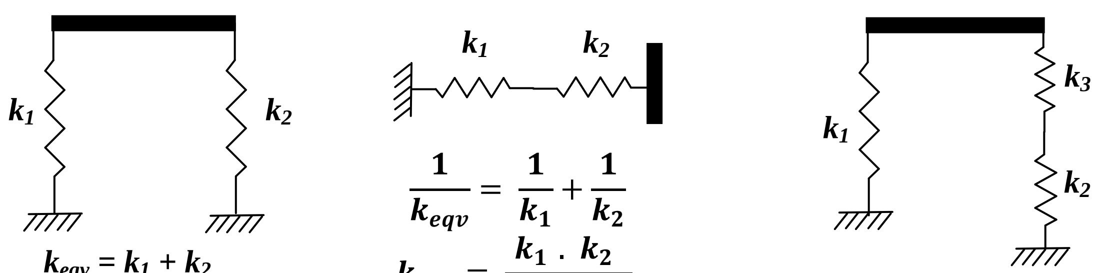
Fig. 3.1 Single Degree of Freedom Systems

STIFFNESS, *K*

Stiffness is the force required to produce a unit displacement in the direction of the force (kips/in, lbs/in).

$$K = F/\Delta$$

When there are multiple columns (members with individual stiffness), the total equivalent stiffness of the SDOF system can be summed from individual member stiffness either in parallel or in series or a combination thereof.



$$k_{eqv} = k_1 + k_2$$

$$\frac{1}{k_{eqv}} = \frac{1}{k_1} + \frac{1}{k_2}$$

$$k_{eqv} = \frac{k_1 \,.\, k_2}{k_1 + k_2}$$

$$k_{eqv} = k_1 + \frac{k_2 k_3}{k_2 + k_3}$$

Fig. 3.2 In-parallel and In-Series Equivalent Stiffness

The inverse of stiffness is flexibility. Thus, it is the deflection produced by a unit force (in/kips, in/lbs).

MEMBER STIFFNESS (MEMBER RIGIDITY)

The member stiffness is a function of the length of the member, ***L***, second moment of inertia, ***I***, Material's Young modulus of elasticity, ***E***, Member's cross sectional area, ***A*** (only in the case of axial stiffness), and ends condition (free, pinned, fixed). Member stiffness is also termed member rigidity. **Table 3.1** summarizes stiffness expressions for different end conditions. The member's maximum deflection equations (Force/stiffness) are also shown in the Table.

$$\textit{stiffness (rigidity)} = \textit{force} \,/\, \textit{deflection}$$

NATURAL PERIOD

If the mass of the SDOF system shown on Fig 3.1.d is subjected to an initial displacement and then released, free vibration occurs about the static position producing a harmonic sinusoidal wave (Fig. 3.3). The time required to complete a full cycle of vibration is the natural period of vibration, ***T***. It is the time between two successive peaks or valleys as shown on Fig. 3.3. Natural period is sometimes referred to as the *fundamental period*.

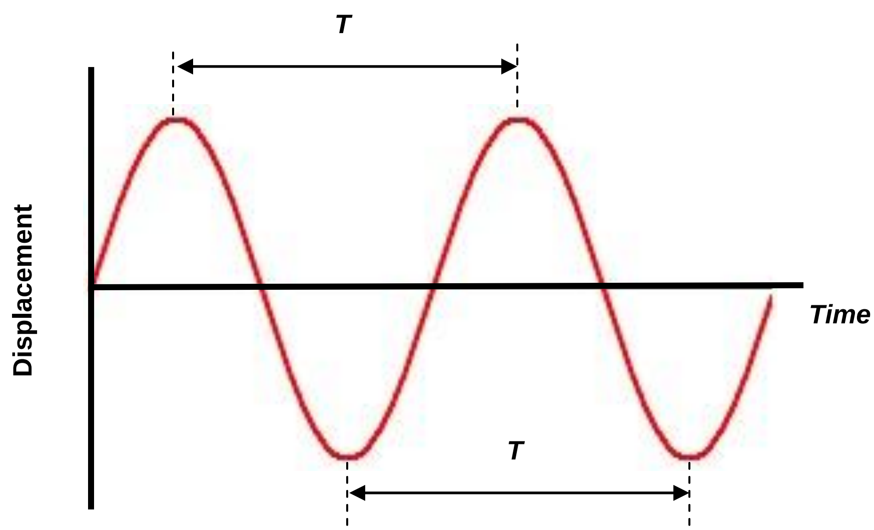


Fig. 3.3 Natural Period of Vibration

Natural period of vibration, T, in sec.:

$$T = 2\pi\sqrt{m/k} \quad \text{(sec)}$$

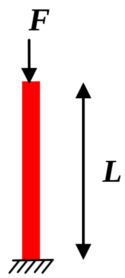
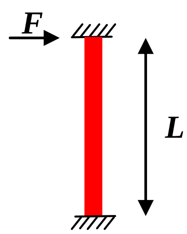
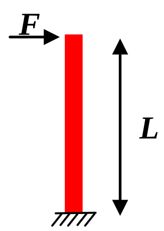
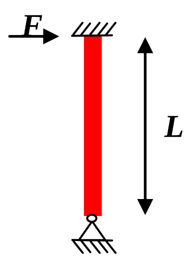
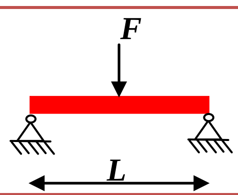
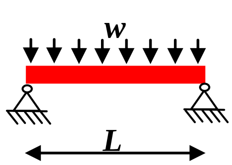
Where $m = W/g$

W = weight of the structure

g = gravitational acceleration = 32.2 ft/sec^2 = 386 in/sec^2

- ➢ The form of this expression indicates that the natural period increases as the mass of the system increases. The natural period also increases as the stiffness of the system decreases.

TABLE 3.1 Member Stiffness for Different Boundary Conditions

MEMBER	*STIFFNESS*	*DEFLECTION*
	$\frac{E\,A}{L}$	$\frac{F\,L}{E\,A}$
	$\frac{12EI}{L^3}$	$\frac{F\,L^3}{12\,E\,I}$
	$\frac{3EI}{L^3}$	$\frac{F\,L^3}{12\,E\,I}$
	$\frac{3EI}{L^3}$	$\frac{F\,L^3}{12\,E\,I}$
	$\frac{48EI}{L^3}$	$\frac{F\,L^3}{48\,E\,I}$
	$\frac{384\,EI}{5\,L^3}$	$\frac{5\,w\,L^4}{384\,E\,I}$

NATURAL FREQUENCY

The invert of natural period T is the natural **linear** frequency, ***f***, expressed in Hertz (cycle/second)

$$f = 1/T \quad \text{(Hz)}$$

ANGULAR NATURAL FREQUENCY

The angular or circular natural frequency, ***ω***, (also known as angular velocity) is defined as the rate of change of angular displacement (during rotation). The angular frequency is measured in radians per second and is given by:

$$\omega = 2\pi . f = 2\pi / T = \sqrt{k/m} \quad \textbf{(rad/sec)}$$

STRUCUTRAL DAMPING

The SDOF system shown in Fig 3.1 (d) will oscillate indefinitely, if there is nothing to dampen the harmonic motion. In practice, the internal friction of the system will resist the motion causing the vibration to die out eventually through many cycles of decaying amplitude of vibration.

The frictional resistance, or damping, ***B***, dissipates the energy of the system, primarily through friction.

- Damping is proportional to the velocity of the vibrating system and can take the form of external or internal viscous damping, body friction damping, radiation damping, and hysteretic damping

CRITICAL DAMPING & DAMPING RATIO

Critical damping, $B_{critical}$, is the amount of damping that brings the system to a static position (equilibrium) in the shortest time (see Fig. 3.4). Both ***underdamped*** and ***overdamped*** motions bring the system back to static position after a long time.

- The ratio of the actual damping of the system to the critical damping is termed damping ratio, ***β***.

$$\beta = B / B_{critical},$$

Typical values of damping ratio range from 2% for welded steel structures, 5% for concrete structures, 10% for masonry shear walls, and 15% for wood structures.

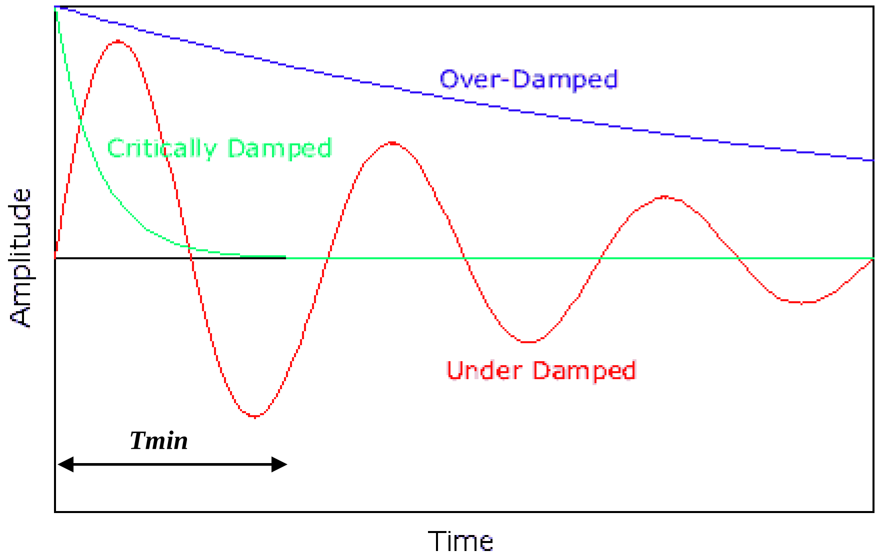


Fig. 3.4 Definition of Critical Damping

It is to be noted that the natural period of vibration of a damped system is approximately the same as a natural period of vibration for free vibration system (undamped vibration), for the typical range of damping ratio in structural applications.

MULTIPLE DEGREE OF FREEDOM SYSTEMS (MDOF)

The multi story structure shown in Fig 3.5 (a) may be idealized by assuming the mass of each floor is lumped at the floor and roof diaphragm.

When assuming a rigid diaphragms and inextensible columns, the multi story building is modeled as a shear building with lateral displacement of the lumped masses as the only degrees of freedom.

The dynamic response of the system is represented by the lateral displacement of the lumped masses with the number of degrees of dynamic freedom, or modes of vibration, ***n***, being equal to the number of masses.

The resultant vibration of the system is given by the superposition of the vibrations of each lumped mass. Each individual mode of vibration has its own period and may be represented by a single degree of freedom system of the same period (See Fig. 3.5). Also, each mode shape remains of constant relative shape regardless of the amplitude of displacement.

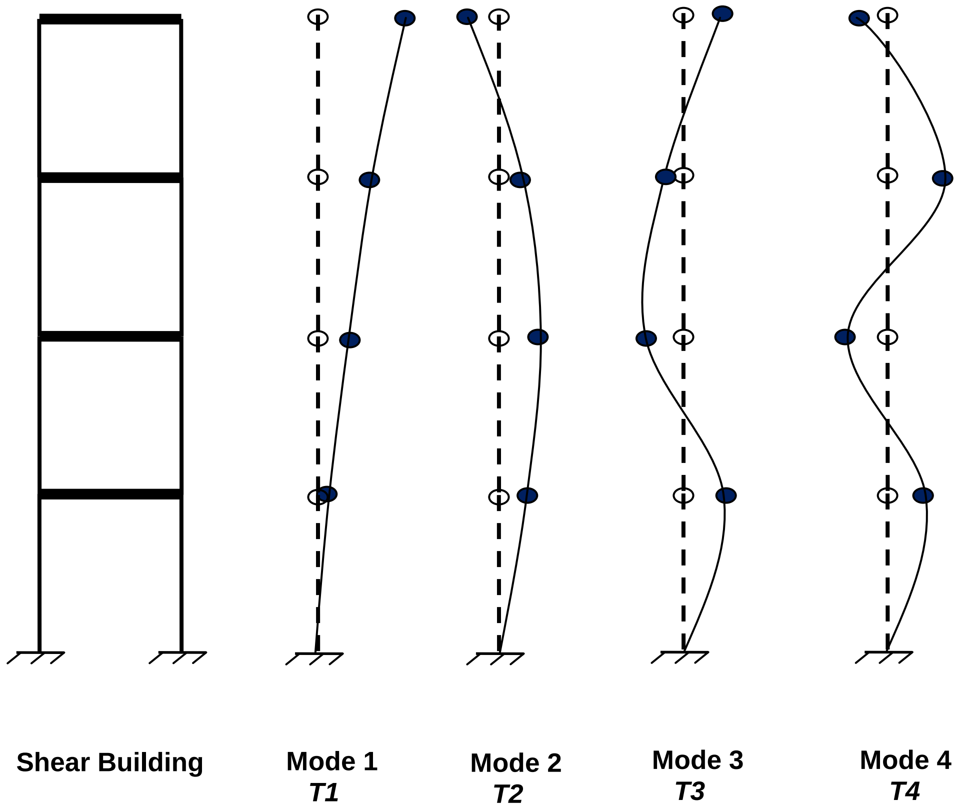


Fig. 3.5 Multiple Degrees of Freedom System

Figure 3.5 shows the 4 modes of vibration of the 4 story shear building.

- The mode of vibration with the longest period (lowest frequency) is termed the first fundamental mode (mode 1)
- The other modes with shorter period of vibration (higher frequencies) are termed higher modes.

A modal analysis procedure is used to determine the dynamic response of a multiple degree of freedom structure. The maximum response for the separate modes is determined by modeling each mode as an individual single degree of freedom system. As the maximum values cannot all occur simultaneously, the maximum values are combined statistically in order o obtain the total response. The square root of the sum of squares is one method for combining the different modes maximums.

It is to be noted that the higher modes does not contribute significantly to the total response. An acceptable procedure is to utilize sufficient modes to ensure that 90% of the participating mass of the structure is included in the analysis. Typically, only few modes are sufficient to obtain the total dynamic response.

RESPONSE SPECTRA

The response of a structure or a SDOF system (as shown on Fig. 3.1) to an earthquake ground motion depends on the natural period of vibration of the structure and its damping ratio.

- The maximum acceleration experienced by the structure due to a specific ground motion is known as ***Spectral Acceleration***, ***Sa.***
- The maximum velocity experienced by the structure due to a specific ground motion is known as ***Spectral Velocity***, ***Sv.***
- The maximum displacement experienced by the structure due to a specific ground motion is known as ***Spectral Displacement***, ***Sd.***

The three spectral responses are interrelated using Newton's Second Law:

$$m \,.\, Sa = k \,.\, Sd$$

Since $\omega^2 = k / m$

$$Sa = \omega^2 \,.\, Sd$$

and

$$Sv = \omega \,.\, Sd$$

$$\boxed{Sa = \omega \,.\, Sv = \omega^2 \,.\, Sd}$$ (undamped system)

The three spectral values represent the maximum response for a SDOF with a certain mass, stiffness and hence natural period, T. By varying the stiffness of the SDOF system and hence the natural period, a plot of any of the three spectral values against various natural periods can be obtained due to a specific ground motion and a certain damping ratio. The resulting plot summarizes the peak (spectral) response of all possible SDOF systems to a particular ground motion, and is called ***response spectrum***. Figure 3.6 shows an example acceleration, velocity and displacement response spectrum for a structure with 5% damping ratio.

- Response spectrum is a plot of all maximum responses (acceleration/ velocity/ displacement) of a structure with certain damping ratio to a particular ground motion.

Original response spectrum curves have random irregularities that could cause large variations in response for a slight change in period. For this reason, response spectrum are often smoothed (idealized) as shown on Fig. 3.7.

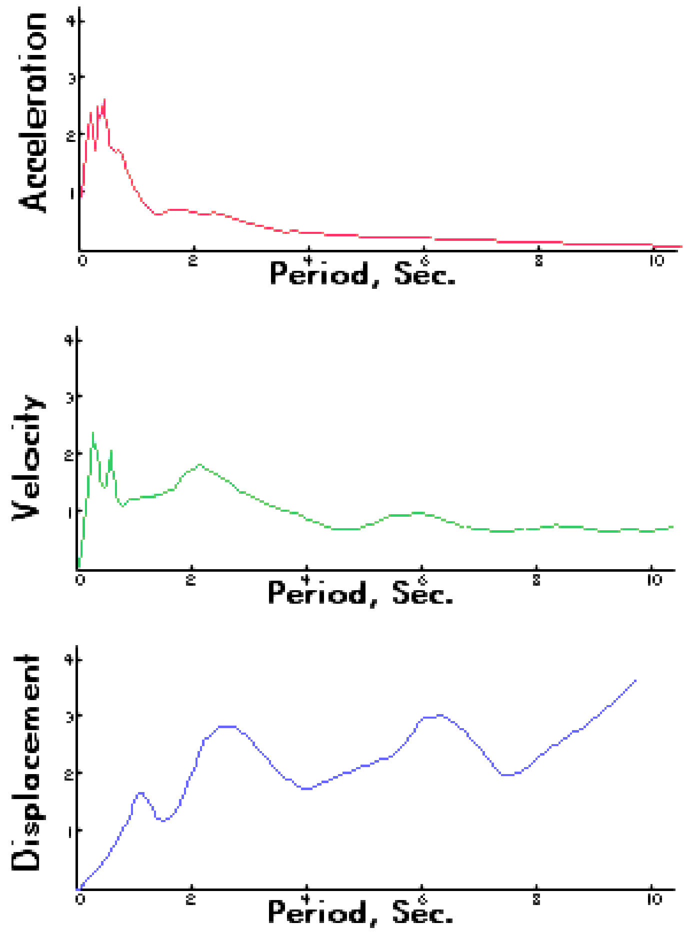


Fig. 3.6   Acceleration, Velocity and Displacement Response Spectrum

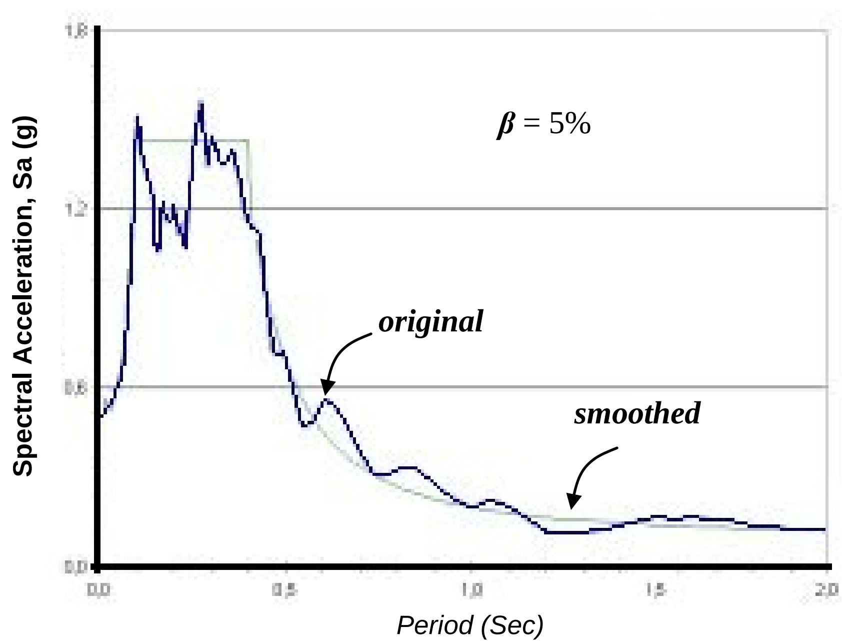


Fig. 3.7   Smoothed Acceleration Response Spectrum

A ***response spectra***, is a collection of ***response spectrum*** for variety of damping ratios and/or soil types. Figure 3.8 shows different acceleration response spectrum for variety of damping ratios. It is evident that by increasing the structural damping, the spectral values decrease.

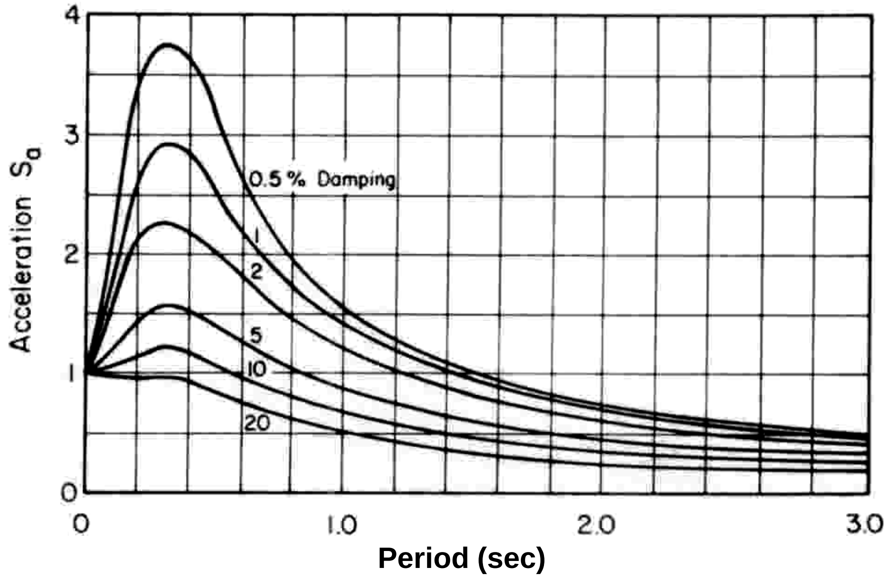


Fig. 3.8 Acceleration Response Spectra for Variety of Damping ratios

In addition to the damping ratio, soil conditions affect significantly the response of structures. Figure 3.9 shows acceleration response spectra for different soil conditions.

- The softer the soil, the longer the period of the structure that experiences the maximum acceleration response.

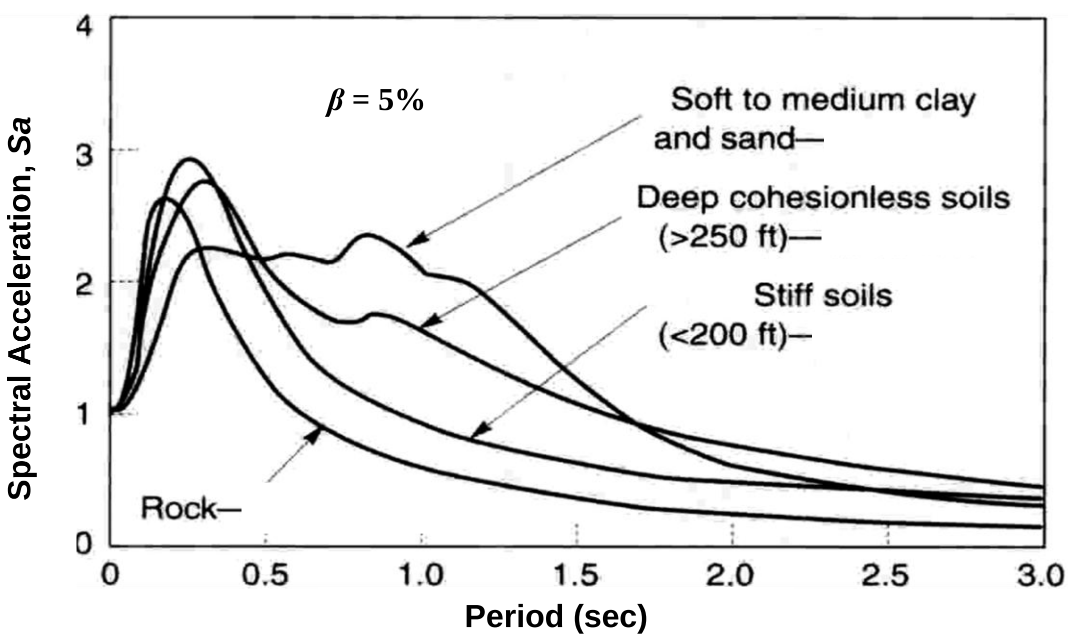


Fig. 3.9 Response Spectra for Different Soil Conditions

NORMALIZED DESIGN RESPONSE SPECTRA

The response spectra shown previously were obtained for a ground motion specific to a particular accelerogram. For design purposes, the response curve must be representative of the characteristics of all seismic events that may occur at a particular site. Thus, several spectrum are developed for the site resulting from the different ground motions expected. The resulting spectra are then averaged and normalized with respect to the effective peak ground acceleration. The average curve is smoothed to eliminate any random irregularities (peaks and valleys).

Since soil conditions at a site substantially affect spectral shape, separate response curves are required for each representative soil type as shown on Fig. 3.10.

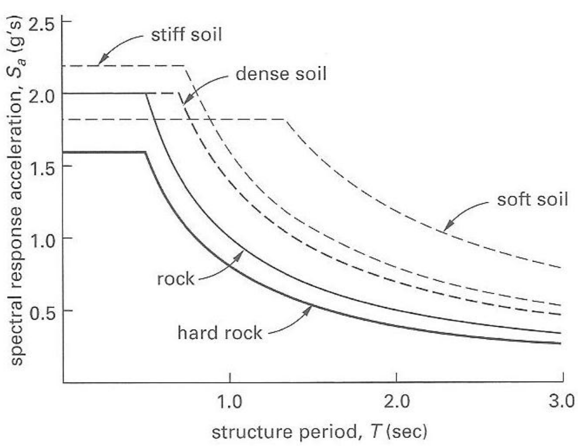


Fig. 3.10 Normalized Design Response Spectra for Different Soil Types

BASE SHEAR

In summary, the maximum force imparted to the structure from an earthquake, ***Base Shear***, ***V***, is given by:

$$V = m \cdot Sa = W \cdot Sa / g$$

The base shear, ***V***, can also be defined in the form of a coefficient/fraction, *C*, of the weight of the structure:

$$V = C \cdot W$$

It is to be noted that the above spectral relationship are approximate for damped system and for MDOF systems.

Example 3.1:

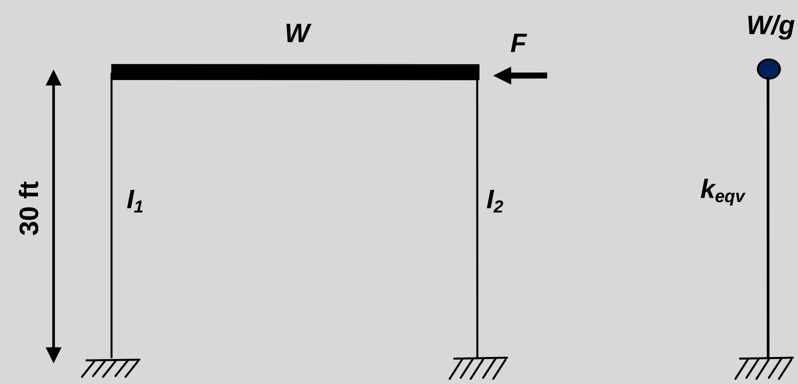
Figure shows a SDOF system, and the Design Response Spectrum where:

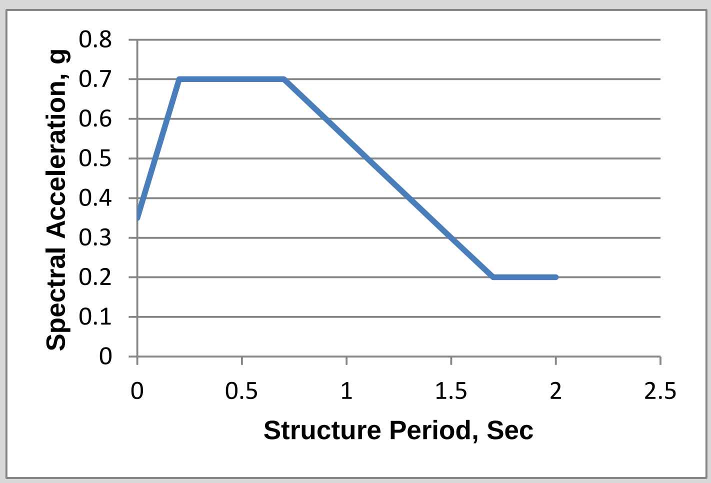
W = 860 kips

E = *29,000* ksi

$I_1 = I_2$ = 22000 k/in

S = 2800 in^3





Design Response Spectrum

Determine Equivalent Stiffness of the SDOF system?

Structure shown can be assumed as SDOF system with lumped mass, m = W/g, and equivalent stiffness k_{eqv}.

$\boldsymbol{k_{eqv}} = 12EI_1/H^3 + 12EI_2/H^3 = 12*29000*22000/(30*12)^3 + 12*29000*22000/(30*12)^3$ = 328 k/in.

Determine natural period and natural frequency of the SDOF system?

Natural period, $T = 2\pi \sqrt{(W/k_{eqv}.g)} = 2*3.14*\sqrt{(860/328.386)} = 0.52$ sec.

Natural frequency, $f = 1/T = 1/0.52 = 1.93$ Hz.

Determine Base shear V of the SDOF system?

From design response spectrum shown, for T=0.52 sec., $S_a = 0.7g$

Base Shear, $V = m*S_a = W*S_a/g = 860*0.7g / g = 602$ kips

Determine lateral deflection Δ of the SDOF system?

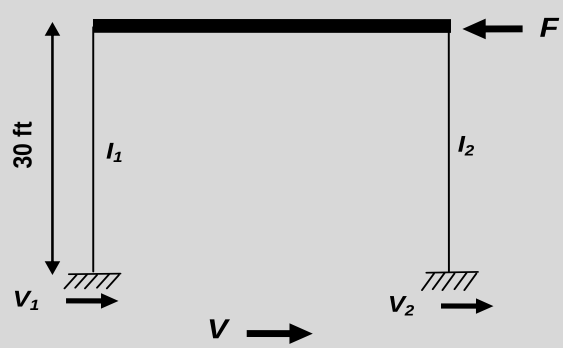
The total base shear V determined above (602 k) equals the lateral force, ***F***, applied at the lumped mass.

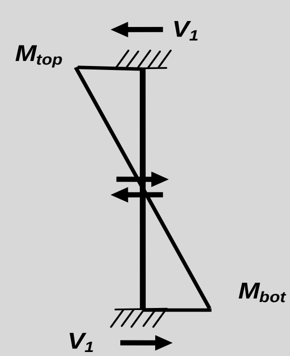
Deflection, $\Delta = F/k_{eqv} = 602/328 = 1.83$ in.

Determine shear force in Column 1?

Base shear will be divided between columns based on their relative stiffness. Since $I_1 = I_2$. Thus $V_1 = V_2 = 602/2 = 301$ kips.

Determine maximum bending moment in Column 1?





$M_{top} = M_{bot} = V_1 * H/2 = 301*30/2 = 4515$ k.ft

Determine maximum bending stress in Column 1?

Bending Stress at bottom (or top), $\sigma = M_{bot}/S = 4515/2800 = 19.4$ ksi

Example 3.2:

Figure shows two SDOF systems, and their Design Response Spectrum where:

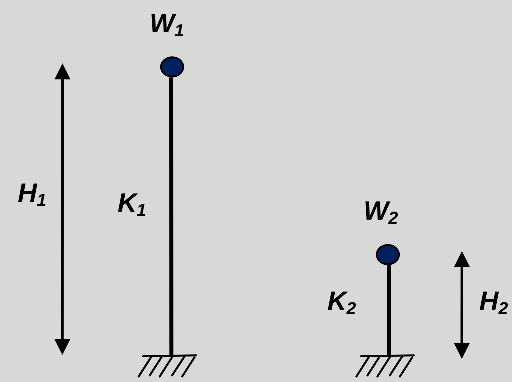
$W_1 = 2 W_2$

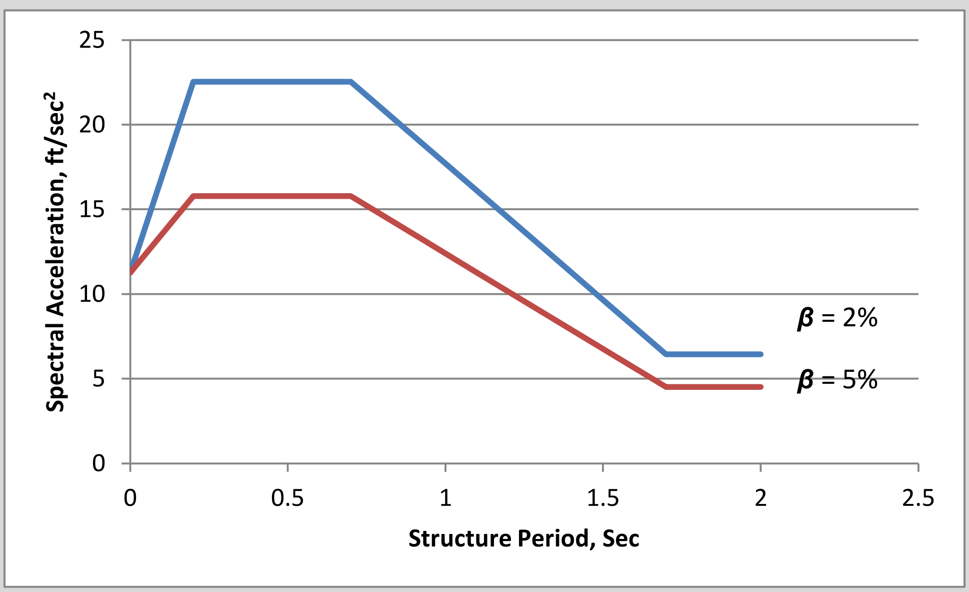
$E_1 = E_2 = E$

$I_1 = 4 I_2$

$H_1 = 3 H_2$

$\beta_1 = 5\%$ $\beta_2 = 2\%$





Design Response Spectrum

Determine the stiffness ratio of SDOF 2 to SDOF 1 ?

$\boldsymbol{K_2}$ = $3*E*I_2/ (H_2)^3$

$\boldsymbol{K_1}$ = $3*E*I_1/(H_1)^3 = 3*E*4*I_2/ (3*H_2)^3 = 3*4*27*E*I_2/ (27*H_2^3) = 0.44$ $\boldsymbol{K_2}$

$\boldsymbol{K_2}$ = 1/0.44 $\boldsymbol{K_1}$ = 2.25 $\boldsymbol{K_1}$

(The shorter the structure, the stiffer it gets)

Determine the natural period ratio of SDOF 2 to SDOF 1 ?

$\boldsymbol{T_1} = 2\pi \sqrt{(W_1/k_1.g)} = 2\pi \sqrt{(2*W_2/(0.44*k_2.g))} = 2\pi * \sqrt{(2*2.25)} * \sqrt{(W_2/k_2.g)}$

$\boldsymbol{T_2} = 2\pi \sqrt{(W_2/k_2.g)}$

$\boldsymbol{T_1} = \sqrt{(2*2.25)} * \boldsymbol{T_2} = 2.21 * \boldsymbol{T_2}$ $\boldsymbol{T_2} = 0.47 * \boldsymbol{T_1}$

(the taller the structure, the longer the natural period)

Determine the base shear ratio of SDOF 2 to SDOF 1 ?

Assume T_1 = 1.0 sec

From Design Response Spectrum

SDOF 1, β_1 = 2% T_1 = 1.0 sec, S_{a1} = 17.6 ft/sec^2

SDOF 2, β_1 = 5% T_2 = 0.47 sec, S_{a2} = 15.8 ft/sec^2

$\boldsymbol{V_1} = W_1 * S_{a1} /g = W_1 * 17.6 /32.2 = 0.55 * W_1 = 0.55 * 2 * W_2 = 1.1 * W_2$

$\boldsymbol{V_2} = W_2 * S_{a2} /g = W_2 * 15.8 /32.2 = 0.49 * W_2$

$\boldsymbol{V_2 / V_1}$ = 0.49 / 1.1 = 0.45 $\boldsymbol{V_2} = 0.45\ \boldsymbol{V_1}$

(generally, the smaller the mass, the less base shear)

MULITPLE CHOICE QUESTIONS

3.1 Member's rigidity can be best described as?

A the inverse of member's deflection

B the inverse of member's stiffness

C the inverse of member's ductility

D the inverse of member's damping

3.2 For a SDOF system, when the mass, *m*, increases, what is the effect on natural period, *T*, and base shear, *V* ?

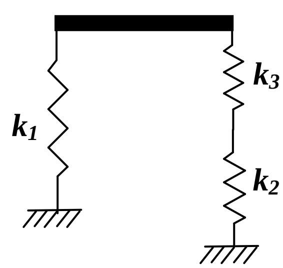
A *T* decreases and *V* increases

B *T* decreases and *V* decreases

C *T* increases and *V* increases

D *T* increases and *V* decreases

3.3 Determine the equivalent stiffness of the SDOF system shown in the figure? Where k_1 = 80 kips/in, k_2 = 60 kips/in, and k_3= 40 kips/in.



A 27 kips/in

B 34 kips/in

C 104 kips/in

D 180 kips/in

3.4 What is the spectral velocity of a single degree of freedom system?

A the minimum velocity experienced by the structure due to a specific ground motion

B the average velocity experienced by the structure due to a specific ground motion

C the maximum acceleration experienced by the structure due to a specific ground motion multiplied times angular frequency

D the maximum displacement experienced by the structure due to a specific ground motion multiplied times angular frequency

3.5 Maximum acceleration experienced by the building from a specific ground motion is defined as?

A peak ground acceleration

B dynamic acceleration

C spectral acceleration

D maximum building velocity relative to ground acceleration

3.6 Determine the natural period of SDOF 2?

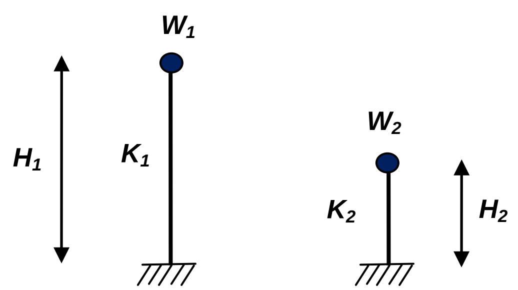
$W_1 = W_2 = W$; $E_1 = E_2 = E$; $I_1 = 2I_2$; $H_1 = 2H_2$

A 0.50 T_1

B 0.85 T_1

C 2.50 T_1

D 3.50 T_1



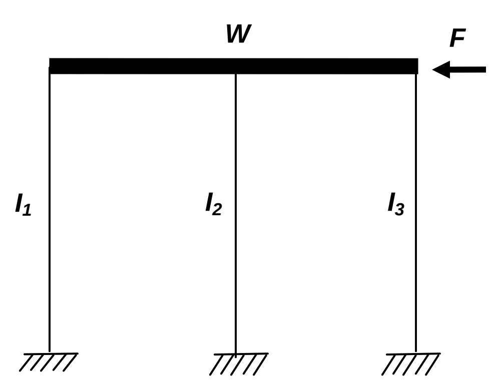
3.7 For the structure shown subjected to lateral force, *F*, determine the ratio of shear forces distributed to the three columns, respectively? The three columns have the same height, modulus of elasticity and boundary conditions. Assume column's second moment of inertia are distributed as $4I_1 = 2I_2 = I_3$.

A 1/7 *F*, 2/7 *F*, 4/7 *F*

B 4/7 *F*, 2/7 *F*, 1/7 *F*

C 1/3 *F*, 1/3 *F*, 1/3 *F*

D -4/6 *F*, -2/6 *F*, 1.0 *F*



3.8 Natural period, *T*, of a SDOF system is determined from:

A the structure response spectrum

B equivalency to the system's linear frequency

C the inverse of the system's angular frequency

D system's mass and stiffness

3.9 What is the damping of an oscillating SDOF system?

A shortest time between successive cycles of vibration

B energy modification response factor

C rate of change of displacement amplitude

D decay of vibration amplitude with time

3.10 For a Multiple Degrees Of Freedom system, MDOF, the term "higher modes" refers to?

A modes of vibration with the longest periods

B modes of vibration with the shortest periods

C modes of vibration with the shortest frequencies

D modes of vibration with the highest participation factor

3.11 Critical damping of an oscillating harmonic system can be best described as the …….?

A damping to bring the harmonic system to static position in the shortest possible time

B ratio of actual damping to critical mass of the system

C decay of vibration amplitude with time

D factor of vibration underdamped amplitude to overdamped amplitude

3.12 Determine the spectral displacement, S_d, of a SDOF system that has natural period of 0.5 sec, and a spectral acceleration, S_a, of 0.75g?

A 1.54 in.

B 1.84 in.

C 2.94 in.

D 4.75 in.

3.13 Natural period of a SDOF system can be best described as?

A the fundamental period of vibration

B the inverse of natural frequency

C the time between two successive peaks or valleys

D all of the above descriptions

3.14 The response spectrum is best described as ……….

A a graph of the maximum responses of SDOF systems to a specified excitation

B the maximum response by a SDOF system to a specified excitation

C a collection of several response spectra

D the maximum ground response of SDOF system to a specified excitation

3.15 The dynamic response of a MDOF system can be best represented by ……..

A the fundamental mode of vibration

B sufficient modes of vibration that contribute more than 90% of the participating mass

C the square root of sum of squares of the first three modes of vibration

D all of the above descriptions

Refer to the building shown in Figure for questions 3.16 to 3.20

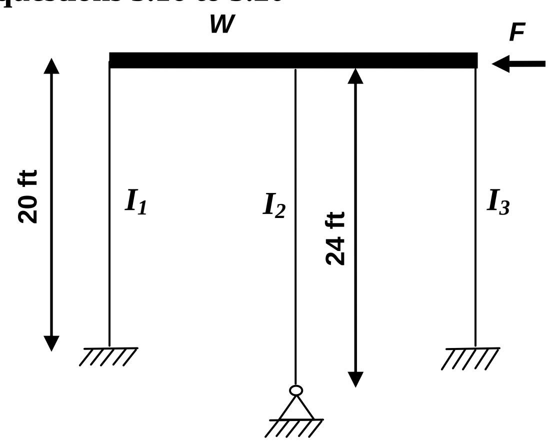
W = 1000 kips

E = 29000 ksi

$I_1 = I_3$ = 12000 in^4

I_2 = 15000 in^4

S_a = 0.75g



3.16 Determine the equivalent stiffness of the SDOF system?

A 360 k/in

B 650 k/in

C 660 k/in

D 825 k/in

3.17 Determine the natural period of the SDOF system?

A 0.35 sec

B 0.39 sec

C 0.40 sec

D 0.54 sec

3.18 Determine the lateral deflection of the SDOF system ?

A 0.91 in

B 1.14 in

C 1.16 in

D 2.10 in

3.19 Determine the shear in Column 2 of the SDOF system ?

A 62 kips

B 249 kips

C 344 kips

D 688 kips

3.20 Determine the flexural stress in Column 2 of the SDOF system, given sectional modulus, S = 1000 in^3 ?

A 12 ksi

B 16 ksi

C 18 ksi

D 36 ksi

3.21, 3.22

TWO STRUCUTRES (similar W, I, different height), find ratio of stiffness, ratio of period T

3.23, 3.24, 3.25

similat to reading height and finding abse shear (2 exams bradard first exam)